

RANDOM WALKS ON DISCRETE ABELIAN GROUPS

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Abstract

In the present paper we find necessary and sufficient conditions for recurrence of random walks on arbitrary subgroups of the group of rational numbers \mathbb{Q} .

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1 Introduction

Let $(\Omega, \mathfrak{A}, P)$ be a probability space, X be a countable discrete Abelian groups, μ be a distribution on X . Recall that a random walk on a group X generated by the distribution μ is a sequence

$$S_n = \xi_1 + \dots + \xi_n, \quad n = 1, 2, \dots,$$

where ξ_j are independent identically distributed with the distribution μ random variables defined on $(\Omega, \mathfrak{A}, P)$ with values on X . The random walk on the group X is said to be recurrent if all elements of X are recurrent i.e. for every $x \in X$ the equality

$$P\{\omega \in \Omega : S_n(\omega) = x \text{ for infinitely many indices } n \in \mathbf{N}\} = 1 \quad (1)$$

holds.

Denote by \mathbb{Z} the additive group of integers, by \mathbb{Q} the additive group of rational numbers considering in the discrete topology, and by $\mathbb{Z}(k)$ the finite cyclic group. R. M. Dudley ([2]) proved that there exists a recurrent random walk on a countable Abelian group X iff X contains no subgroup isomorphic to \mathbb{Z}^3 . Dudley's proof is not constructive and therefore it is natural to look for other effective recurrence conditions.

Such conditions on μ were studied

- (a) on the weak direct product $\mathbb{Z}(2)^{\aleph_0^*}$ ([1]),
- (b) on the weak direct product $\mathbf{P}_{i \in \mathbf{N}}^* \mathbb{Z}(k_i)$ ([4]),
- (c) on the factor group \mathbb{Q}/\mathbb{Z} and its subgroups ([4]),
- (d) on the weak direct product $\mathbb{Z}(k)^{\aleph_0^*}$ ([5]),
- (e) on subgroups $H_p = \left\{ \frac{m}{p^n} : n = 1, 2, \dots; m \in \mathbb{Z} \right\}$ of the group \mathbb{Q} ([5]),
- (f) on groups of the form $\mathbb{Z}^k \times \mathbf{P}_{i \in \mathbf{N}}^* \mathbb{Z}(k_i)$ ([7]).

In the present paper we find necessary and sufficient conditions for recurrence of random walks on arbitrary subgroups of the group of rational numbers \mathbb{Q} .

Let X be a locally compact second countable Abelian group, $Y = X^*$ be its character group, and (x, y) be the value of a character $y \in Y$ at an element $x \in X$. Let

$$\hat{\mu}(y) = \int_X (x, y) d\mu(x)$$

be the characteristic function of a distribution μ on X . Denote by m_X the Haar measure on X .

The following recurrence criterion was proved in [8].

Theorem A ([8]). *Let X be a countable discrete Abelian group. Let μ be a distribution on X . A random walk defined by the distribution μ is recurrent iff*

$$\int_Y \operatorname{Re} \frac{1}{1 - \hat{\mu}(y)} dm_Y = \infty \quad (2)$$

In cases (b), (c), (d), (f) authors of corresponding papers use Theorem A to obtain necessary and sufficient conditions for recurrence of random walks. In these cases the character groups have quite simple structure.

The case (e) is more complicated. Authors of [5] state that Theorem A is useless for obtaining necessary and sufficient conditions for recurrence of random walks on a group X if its character group Y has a complicated structure as, for example, the group $H_p = \left\{ \frac{m}{p^n} : n = 1, 2, \dots; m \in \mathbb{Z} \right\}$. In this case the character group Y is a p -adic solenoid. In [5] necessary and sufficient conditions for recurrence of random walks on a group H_p were obtained without using of Theorem A.

In the present paper we do use Theorem A and find necessary and sufficient conditions for recurrence of random walks on arbitrary subgroups of the group of rational numbers \mathbb{Q} . In this case the character group Y is a a -adic solenoid. Note that the results of the paper [5] for the groups H_p follows directly from our paper.

2 Notation and definitions

In the present paper we use some results from the Pontryagin duality theory (see [6]).

In the paper we consider random walks on arbitrary subgroups of the group of rational numbers not isomorphic to \mathbb{Z} .

Let $a = (a_1, a_2, \dots)$ be a sequence of integers where all $a_j > 1$. Consider a group

$$H_a = \left\{ \frac{m}{a_1 a_2 \dots a_n} : n = 1, 2, \dots; m \in \mathbb{Z} \right\}. \quad (3)$$

It is well-known that any subgroup of the group \mathbb{Q} not isomorphic to \mathbb{Z} has the form (3) for some $a = (a_1, a_2, \dots)$. Particularly, if all $a_j = p$ then we obtain the group of $H_p = \left\{ \frac{m}{p^n} : n = 1, 2, \dots; m \in \mathbb{Z} \right\}$.

In order to apply Theorem A, we have to describe the character group of the group H_a .

Let Δ_a be a group of a -adic integers. Consider the group $\mathbb{R} \times \Delta_a$. Let B the subgroup of $\mathbb{R} \times \Delta_a$ of the form $B = \{(n, n\mathbf{u})\}_{n=-\infty}^{\infty}$, where $\mathbf{u} = (1, 0, 0, \dots, 0, \dots) \in \Delta_a$. The factor group $\Sigma_a = \mathbb{R} \times \Delta_a / B$ is called the a -adic solenoid. The group Σ_a is compact and connected (see [6, §10]). The group Σ_a is topologically isomorphic to the character group of the group H_a (see [6, (25.3)]).

Denote by \mathbb{T} the circle group (the one-dimensional torus), i.e. $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. It is convenient for us to use another consideration of the a -adic solenoid as a subgroup of the infinite-dimensional torus $\mathbb{T}^{\mathbb{N}_0}$.

Consider the mapping $f : \mathbb{R} \times \Delta_a \longrightarrow \mathbb{T}^{\mathbb{N}_0}$, defined by

$$f(t, \mathbf{y}) \mapsto z = (z_1, z_2, \dots), \quad z_j = \exp \left(\frac{2\pi i}{a_1 \cdots a_j} (t - (y_0 + a_1 y_1 + \dots + a_1 a_2 \dots a_{j-1} y_{j-1})) \right),$$

where $t \in \mathbb{R}$, $\mathbf{y} = (y_0, y_1, \dots) \in \Delta_a$.

It is not difficult to verify that f is a continuous homomorphism, $\text{Ker} f = B$ and $G = \text{Im} f = \{z = (z_1, z_2, \dots, z_n, \dots) \in \mathbb{T}^{\mathbb{N}_0} : z_k^{a_k} = z_{k-1}\}$ is a closed subgroup of the infinite-dimensional torus $\mathbb{T}^{\mathbb{N}_0}$. Then $G \cong \Sigma_a$.

The consideration of the a -adic solenoid Σ_a as the subgroup

$$G = \{z = (z_1, z_2, \dots, z_n, \dots) \in \mathbb{T}^{\mathbb{N}_0} : z_k^{a_k} = z_{k-1}\}$$

allows us to verify easily the following: if $h = \frac{m}{a_1 a_2 \dots a_n}$ is a character of the group Σ_a , then $(z, h) = z_{n+1}^m$, $z = (z_1, z_2, \dots, z_n, \dots) \in G$.

3 Main results

Let $a = (a_1, a_2, \dots)$ be a sequence of integers where all $a_j > 1$. Consider the group $X = H_a$. Note that numbers

$$e_{\pm 0} = \pm 1, \quad e_{\pm 1} = \pm \frac{1}{a_1}, \quad e_{\pm 2} = \pm \frac{1}{a_1 a_2}, \quad \dots, \quad e_{\pm n} = \pm \frac{1}{a_1 \dots a_n}, \quad \dots$$

are natural generators of the group X .

We consider on X a distribution μ of the form

$$\mu\{e_{\pm j}\} = \frac{q_j}{2}, \quad \sum_{j=0}^{\infty} q_j = 1, \quad q_j \geq 0. \quad (4)$$

The distribution μ defines a random walk on X .

For a compact group Y we suppose that a Haar measure m_Y is normalized in such a way that $m_Y(Y) = 1$.

In the following theorem we obtain sufficient conditions for recurrence of a random walk defined by a distribution μ on the group $X = H_a$.

Theorem 1. *Let $X = H_a$, where $a = (a_1, a_2, \dots)$, $a_j > 1$. Let μ be a distribution on X of the form (4). Consider on X a random walk defined by μ . The condition*

$$\sum_{n=1}^{\infty} \frac{1}{a_1 \dots a_n \sqrt{q_n + q_{n+1} + \dots}} = \infty \quad (5)$$

is sufficient for recurrence of a random walk defined by μ on X .

Proof. The proof of Theorem 1 is based on Theorem A. The character group of the group X is topologically isomorphic to the group Σ_a . In order not to complicate the notation, we will assume that $Y = \Sigma_a$. It is convenient to consider the realization of the a -adic solenoid as a subgroup in $\mathbb{T}^{\mathbb{N}_0}$. Then each element of Y is a sequence $y = (y_1, y_2, \dots)$, where $y_n \in \mathbb{T}$, $y_n^{a_n} = y_{n-1}$. Put $y_n = e^{ib_n}$. Thus the sequence (y_1, y_2, \dots) corresponds to a sequence (b_1, b_2, \dots) .

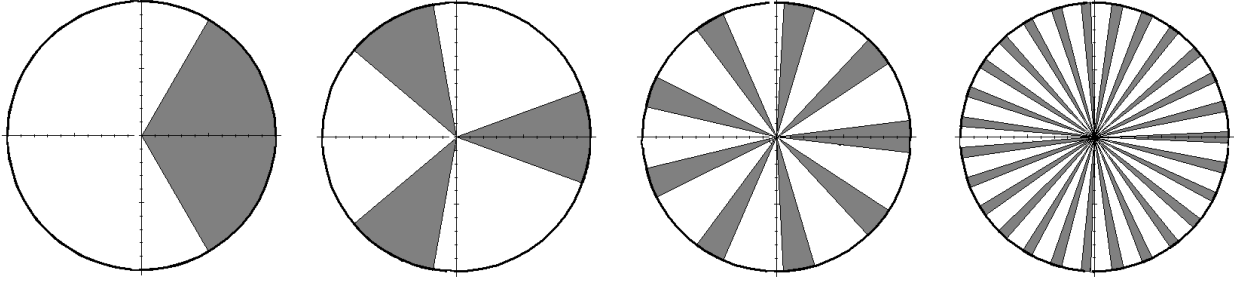


Figure 1: The domains of variation of elements b_1, b_2, b_3, b_4 respectively in the subset $Y \setminus E_0$ for the case when each $a_j = 3$.

Numbers b_n are defined modulo 2π . We will take these numbers either in the interval $[0, 2\pi)$ or in the interval $[-\pi, \pi)$ depending on how it is convenient for us. Regardless of an interval in which we take numbers b_n , misunderstandings will not arise. Note that

$$a_{n+1}b_{n+1} = b_n \pmod{2\pi}. \quad (6)$$

Note also that $(e_{\pm j}, y) = e^{\pm i b_{j+1}} = \cos b_{j+1} \pm i \sin b_{j+1}$. Then

$$\hat{\mu}(y) = \int_X (x, y) d\mu(x) = \frac{1}{2} \sum_{j=0}^{\infty} q_j(e_j, y) + \frac{1}{2} \sum_{j=0}^{\infty} q_j(e_{-j}, y) = \sum_{j=0}^{\infty} q_j \cos b_{j+1}. \quad (7)$$

We will build by induction on the group Y a system of non overlapping sets $E_0, E_1, \dots, E_n, \dots$ such that

$$\sum_{n=0}^{\infty} m_Y(E_n) = 1. \quad (8)$$

Put

$$E_0 = \left\{ y \in Y : b_1 \notin \left[-\frac{\pi}{a_1}, \frac{\pi}{a_1} \right] \right\}.$$

Using the invariance of the Haar measure we obtain that

$$m_Y(E_0) = \frac{a_1 - 1}{a_1}. \quad (9)$$

Note

$$\begin{aligned} Y \setminus E_0 = \{ y \in Y : b_1 \in \left[-\frac{\pi}{a_1}, \frac{\pi}{a_1} \right], b_2 \in \left[\frac{2\pi k}{a_2} - \frac{\pi}{a_1 a_2}, \frac{2\pi k}{a_2} + \frac{\pi}{a_1 a_2} \right] (k = 0, 1, \dots, a_2 - 1), \dots \\ \dots, b_n \in \left[\frac{2\pi k}{a_n} - \frac{\pi}{a_1 \dots a_n}, \frac{2\pi k}{a_n} + \frac{\pi}{a_1 \dots a_n} \right] (k = 0, 1, \dots, a_n - 1), \dots \} \end{aligned}$$

See Figure 1 for the case when each $a_j = 3$.

Put

$$\begin{aligned} E_1 = \left\{ y \in Y \setminus E_0 : b_2 \notin \left[-\frac{\pi}{a_1 a_2}, \frac{\pi}{a_1 a_2} \right] \right\} = \\ \left\{ y \in Y \setminus E_0 : b_1 \in \left[-\frac{\pi}{a_1}, \frac{\pi}{a_1} \right], b_2 \in \left[\frac{2\pi k}{a_2} - \frac{\pi}{a_1 a_2}, \frac{2\pi k}{a_2} + \frac{\pi}{a_1 a_2} \right] (k = 1, \dots, a_2 - 1) \right\}. \quad (10) \end{aligned}$$

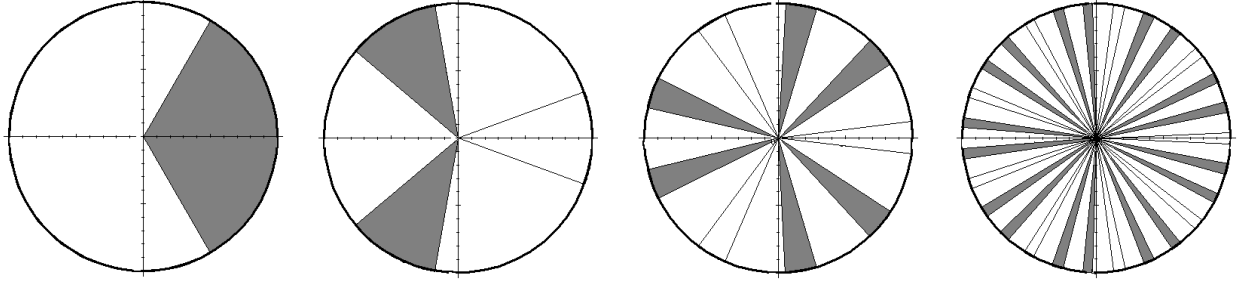


Figure 2: The domains of variation of elements b_1, b_2, b_3, b_4 respectively in the subset E_1 for the case when each $a_j = 3$.

It is easy to see that $m_Y(E_1) = \frac{a_2-1}{a_1 a_2}$ (see Figure 2 for the case when each $a_j = 3$).

We define by induction a sequence of sets

$$E_n = \left\{ y \in Y \setminus \bigcup_{j=0}^{n-1} E_j : b_{n+1} \notin \left[-\frac{\pi}{a_1 \dots a_{n+1}}, \frac{\pi}{a_1 \dots a_{n+1}} \right] \right\}. \quad (11)$$

Actually sets E_n can be defined by means of the coordinate b_{n+1} . For a better understanding we note that

$$\begin{aligned} E_n &= \left\{ y \in Y : b_{n+1} \in \left[\frac{2\pi k}{a_{n+1}} - \frac{\pi}{a_1 a_2 \dots a_{n+1}}, \frac{2\pi k}{a_{n+1}} + \frac{\pi}{a_1 a_2 \dots a_{n+1}} \right] (k = 1, \dots, a_{n+1} - 1) \right\} = \\ &= \left\{ y \in Y \setminus \bigcup_{j=0}^{n-1} E_j : b_1 \in \left[-\frac{\pi}{a_1}, \frac{\pi}{a_1} \right], b_2 \in \left[-\frac{\pi}{a_1 a_2}, \frac{\pi}{a_1 a_2} \right], \dots, b_n \in \left[-\frac{\pi}{a_1 a_2 \dots a_n}, \frac{\pi}{a_1 a_2 \dots a_n} \right], \right. \\ &\quad \left. b_{n+1} \in \left[\frac{2\pi k}{a_{n+1}} - \frac{\pi}{a_1 a_2 \dots a_{n+1}}, \frac{2\pi k}{a_{n+1}} + \frac{\pi}{a_1 a_2 \dots a_{n+1}} \right] (k = 1, \dots, a_{n+1} - 1) \right\}. \end{aligned} \quad (12)$$

See Figure 3 for the case when each $a_j = 3$.

Using (11) and invariance of the Haar measure it is easy to verify by induction that

$$m_Y(E_n) = \frac{a_{n+1} - 1}{a_1 \dots a_{n+1}} \quad (13)$$

for all $n = 0, 1, 2, \dots$

By construction, we have obtained that $E_i \cap E_j = \emptyset$ for $i \neq j$. We have

$$\sum_{n=0}^{\infty} m_Y(E_n) = \frac{a_1 - 1}{a_1} + \frac{a_2 - 1}{a_1 a_2} + \frac{a_3 - 1}{a_1 a_2 a_3} + \dots = 1 - \frac{1}{a_1} + \frac{1}{a_1} - \frac{1}{a_1 a_2} + \frac{1}{a_1 a_2} - \frac{1}{a_1 a_2 a_3} + \dots = 1.$$

Let α_n be a number such that $0 < \alpha_n < 1$.

Put

$$A_n = \left\{ y \in E_n : b_1 \in \left(-\frac{\pi \alpha_n}{a_1}, \frac{\pi \alpha_n}{a_1} \right) \right\} = \left\{ y \in E_n : b_2 \in \left(-\frac{\pi \alpha_n}{a_1 a_2}, \frac{\pi \alpha_n}{a_1 a_2} \right) \right\} = \dots$$

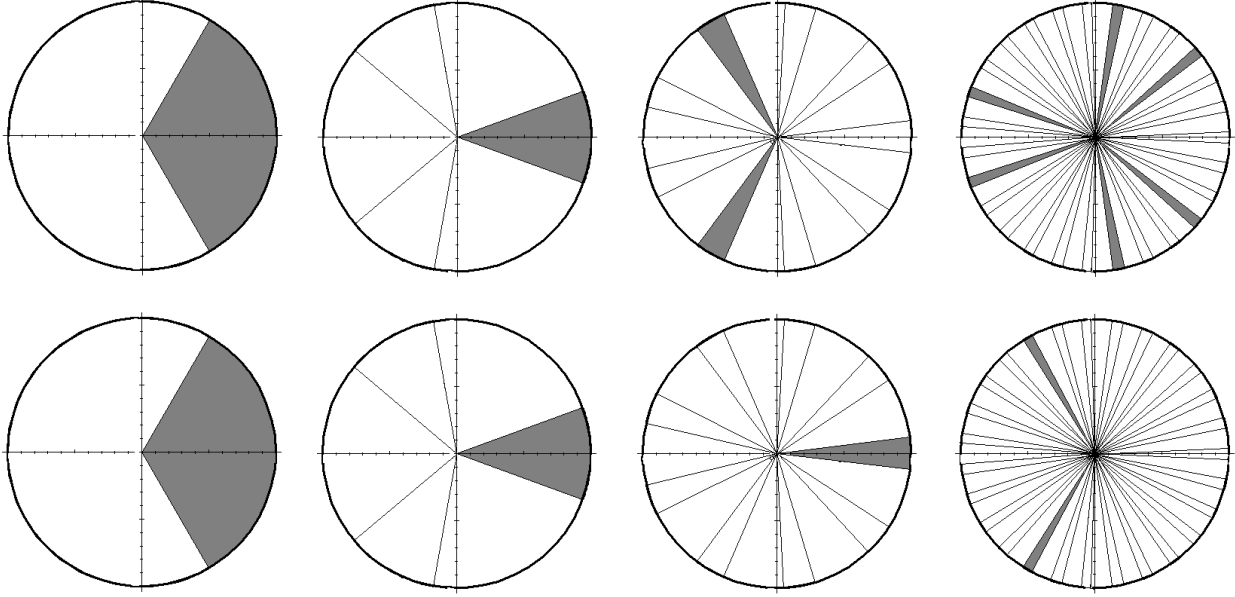


Figure 3: The domains of variation of elements b_1, b_2, b_3, b_4 respectively in the subsets E_2 and E_3 for the case when each $a_j = 3$.

$$= \left\{ y \in E_n : b_k \in \left(-\frac{\pi\alpha_n}{a_1 a_2 \dots a_k}, \frac{\pi\alpha_n}{a_1 a_2 \dots a_k} \right) \right\} = \dots$$

Since the measure of E_n coincides with the measure of projections on the $(n+1)$ circle, it is obvious that

$$m_Y(A_n) = \alpha_n m_Y(E_n) = \alpha_n \frac{a_{n+1} - 1}{a_1 \dots a_{n+1}}. \quad (14)$$

We evaluate from below the sum of the following series

$$\sum_{n=0}^{\infty} \int_{E_n} \frac{1}{1 - \hat{\mu}(y)} dm_Y(y). \quad (15)$$

We have

$$\sum_{n=0}^{\infty} \int_{E_n} \frac{1}{1 - \hat{\mu}(y)} dm_Y(y) \geq \sum_{n=1}^{\infty} \int_{A_n} \frac{1}{1 - \hat{\mu}(y)} dm_Y(y). \quad (16)$$

We evaluate from above $1 - \hat{\mu}(y)$ for $y \in A_n$. We have

$$\begin{aligned} 1 - \hat{\mu}(y) &= 1 - \sum_{j=0}^{\infty} q_j \cos b_{j+1} = \sum_{j=0}^{\infty} q_j (1 - \cos b_{j+1}) \leq \\ &\leq q_0 \left(1 - \cos \frac{\pi\alpha_n}{a_1} \right) + q_1 \left(1 - \cos \frac{\pi\alpha_n}{a_1 a_2} \right) + \dots + q_{n-1} \left(1 - \cos \frac{\pi\alpha_n}{a_1 \dots a_n} \right) + 2(q_n + q_{n+1} + \dots) \end{aligned} \quad (17)$$

Since

$$\frac{2t^2}{\pi^2} \leq 1 - \cos t \leq \frac{t^2}{2}, \quad t \in [0, \frac{\pi}{2}], \quad (18)$$

we can continue evaluation (17) in the following way.

$$\begin{aligned} 1 - \hat{\mu}(y) &\leq \frac{\pi^2 \alpha_n^2}{a_1^2} \left(1 + \frac{1}{a_2^2} + \frac{1}{a_2^2 a_3^2} + \dots + \frac{1}{a_2^2 \dots a_n^2} \right) + 2(q_n + q_{n+1} + \dots) \\ &\leq C_1(\alpha_n^2 + q_n + q_{n+1} + \dots), \end{aligned} \quad (19)$$

where C_1 is a constant which does not depend on n .

Taking into account (19) and (14), we can continue evaluation (16):

$$\begin{aligned} \sum_{n=0}^{\infty} \int_{E_n} \frac{1}{1 - \hat{\mu}(y)} dm_Y(y) &\geq \sum_{n=1}^{\infty} \int_{A_n} \frac{1}{1 - \hat{\mu}(y)} dm_Y(y) \geq C_2 \sum_{n=0}^{\infty} \frac{\alpha_n(a_{n+1} - 1)}{a_1 \dots a_{n+1}(\alpha_n^2 + q_n + q_{n+1} + \dots)} \\ &\geq C_3 \sum_{n=0}^{\infty} \frac{\alpha_n}{a_1 \dots a_n(\alpha_n^2 + q_n + q_{n+1} + \dots)}, \end{aligned} \quad (20)$$

where C_2, C_3 are constants which do not depend on n .

Note that for $a \in (0, 1)$ the function $f(x) = \frac{x}{x^2 + a}$ has a maximum value on $[0, 1]$ at $x = \sqrt{a}$. Taking this into account from (20) we find that

$$\sum_{n=0}^{\infty} \int_{E_n} \frac{1}{1 - \hat{\mu}(y)} dm_Y(y) \geq C_4 \sum_{n=0}^{\infty} \frac{1}{a_1 \dots a_n \sqrt{q_n + q_{n+1} + \dots}}, \quad (21)$$

where C_4 is a constant which does not depend on n .

Now we can prove that condition (5) is sufficient for recurrence of a random walk defined by the distribution μ . Indeed, suppose that condition (5) is satisfied and a random walk defined by the distribution μ is transient. Then Theorem A implies

$$\int_Y \frac{1}{1 - \hat{\mu}(y)} dm(y) < \infty.$$

But then

$$\int_Y \frac{1}{1 - \hat{\mu}(y)} dm(y) = \sum_{n=0}^{\infty} \int_{E_n} \frac{1}{1 - \hat{\mu}(y)} dm(y) < \infty.$$

It follows from (21) that series in (5) converges. ■

Remark 1. If we put in Theorem 1 all $a_j = p$ where p is prime, we obtain Theorem 3 of the article [5] for a random walk defined on the group H_p .

Thus, we have a sufficient condition for the recurrence of a random walk defined on a subgroup of the group of rational numbers \mathbb{Q} , not isomorphic to \mathbb{Z} . In the following theorem we obtain a necessary condition for the recurrence of a random walk defined on a subgroup of a group of rational numbers \mathbb{Q} .

We need the following well-known property of characteristic functions (see e.g. [3, §2]).

Lemma 1. *Let X be a second countable locally compact Abelian group, let μ be a distribution on X . The following conditions are equivalent.*

- (i) *The support of the distribution μ is not contained in any coset of some subgroup in X .*
- (ii) $\{y \in Y : |\hat{\mu}(y)| = 1\} = \{0\}$.

Theorem 2. Let $X = H_a$, where $a = (a_1, a_2, \dots)$, $a_j > 1$. Let μ be a distribution on X of the form (4). We suppose that in (4) all $q_j > 0$. The condition

$$\sum_{n=1}^{\infty} \frac{a_{n+1}}{a_1 \dots a_n \sqrt{q_n}} = \infty \quad (22)$$

is necessary for recurrence of a random walk defined by μ on X .

Proof. Let sets E_n be such as in Theorem 1. We evaluate from above the sum of series (15).

Note that the support of the distribution μ is not contained in any coset of some subgroup in X . Then Lemma 1 implies that

$$\{y \in Y : |\hat{\mu}(y)| < 1\} = \{0\}. \quad (23)$$

Therefore

$$\frac{1}{1 - \hat{\mu}(y)} = \sum_{k=0}^{\infty} \hat{\mu}^k(y), \quad y \neq 0.$$

By the Lebesgue-Levi theorem we have

$$\begin{aligned} \sum_{n=0}^{\infty} \int_{E_n} \frac{1}{1 - \hat{\mu}(y)} dm_Y(y) &= \sum_{n=0}^{\infty} \int_{E_n} \sum_{k=0}^{\infty} \hat{\mu}(y)^k dm_Y(y) = \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \int_{E_n} \hat{\mu}(y)^k dm_Y(y). \end{aligned} \quad (24)$$

It follows from (7) that $\hat{\mu}(y) \leq q_0 \cos \frac{\pi}{a_1} + q_1 + q_2 + \dots = 1 - q_0(1 - \cos \frac{\pi}{a_1})$ for $y \in E_0$. Hence

$$\int_{E_0} \frac{1}{1 - \hat{\mu}(y)} dm(y) < \frac{a_1 - 1}{a_1 q_0 (1 - \cos \frac{\pi}{a_1})}.$$

Besides taking into account (13), we note that

$$\sum_{n=1}^{\infty} \int_{E_n} dm(y) = \frac{1}{a_1}.$$

Hence

$$\sum_{n=0}^{\infty} \int_{E_n} \sum_{k=0}^{\infty} \hat{\mu}(y)^k dm_Y(y) = C_1 + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_{E_n} \hat{\mu}(y)^k dm_Y(y), \quad (25)$$

where C_1 is a constant which does not depend on n .

We evaluate from above $\hat{\mu}(y)$ for $y \in E_n$. It is obvious that

$$\begin{aligned} \hat{\mu}(y) &= \sum_{j=0}^{\infty} q_j \cos b_{j+1} \leq \sum_{j=0}^n q_j \cos b_{j+1} + q_{n+1} + q_{n+2} + \dots \\ &= \sum_{j=0}^n q_j \cos b_{j+1} + 1 - q_0 - \dots - q_n = 1 - \sum_{j=0}^n q_j (1 - \cos b_{j+1}). \end{aligned} \quad (26)$$

Since $y \in E_n$, it follows from (11) that $b_{n+1} \in \left[\frac{2\pi k}{a_{n+1}} - \frac{\pi}{a_1 \dots a_{n+1}}, \frac{2\pi k}{a_{n+1}} + \frac{\pi}{a_1 \dots a_{n+1}} \right]$ ($k = 1, \dots, a_{n+1} - 1$). Hence for any n the inequality

$$1 - \cos b_{n+1} \geq 1 - \cos \left(\frac{2\pi}{a_{n+1}} - \frac{\pi}{a_1 \dots a_{n+1}} \right) \quad (27)$$

is fulfilled. If $a_{n+1} = 2$ or $a_{n+1} = 3$ then $\frac{\pi}{2} < \frac{2\pi}{a_{n+1}} - \frac{\pi}{a_1 \dots a_{n+1}} < \pi$. Hence, taking into account (18) and (27), we have

$$1 - \cos b_{n+1} \geq 1 > \frac{2}{a_{n+1}^2}. \quad (28)$$

If $a_{n+1} > 3$ then $0 < \frac{2\pi}{a_{n+1}} - \frac{\pi}{a_1 \dots a_{n+1}} < \frac{\pi}{2}$. Hence, taking into account (27) and (18), we have

$$1 - \cos b_{n+1} \geq \frac{2}{\pi^2} \left(\frac{2\pi}{a_{n+1}} - \frac{\pi}{a_1 \dots a_{n+1}} \right)^2 \geq \frac{2}{a_{n+1}^2}. \quad (29)$$

Recall also that $b_n \in \left[-\frac{\pi}{a_1 \dots a_n}, \frac{\pi}{a_1 \dots a_n} \right]$ for $y \in E_n$. Putting $t = b_n$ and taking into account (6), we can rewrite estimate (26) in the following way

$$\hat{\mu}(y) \leq 1 - q_0(1 - \cos a_2 \dots a_n t) - q_1(1 - \cos a_3 \dots a_n t) - \dots - q_{n-2}(1 - \cos a_n t) - q_{n-1}(1 - \cos t) - C_2 \frac{q_n}{a_{n+1}^2}, \quad (30)$$

where $t \in \left[-\frac{\pi}{a_1 \dots a_n}, \frac{\pi}{a_1 \dots a_n} \right]$.

Put $\tilde{t} = a_2 \dots a_n t$. Then $\tilde{t} \in \left[-\frac{\pi}{a_1}, \frac{\pi}{a_1} \right]$. We can rewrite estimate (30) in the following way

$$\hat{\mu}(y) \leq 1 - q_0(1 - \cos \tilde{t}) - q_1 \left(1 - \cos \frac{\tilde{t}}{a_2} \right) - \dots - q_{n-2} \left(1 - \cos \frac{\tilde{t}}{a_2 \dots a_{n-1}} \right) - q_{n-1} \left(1 - \cos \frac{\tilde{t}}{a_2 \dots a_n} \right) - C_2 \frac{q_n}{a_{n+1}^2}, \quad (31)$$

where $\tilde{t} \in \left[-\frac{\pi}{a_1}, \frac{\pi}{a_1} \right]$.

Taking into account (18) we obtain

$$1 - \cos \frac{\tilde{t}}{a_2 \dots a_j} \geq \frac{2\tilde{t}^2}{\pi^2 a_2^2 \dots a_j^2}.$$

Thus we can continue inequality (31)

$$\hat{\mu}(y) \leq 1 - \frac{2\tilde{t}^2}{\pi^2} \left(q_0 + \frac{q_1}{a_2^2} + \frac{q_2}{a_2^2 a_3^2} + \dots + \frac{q_{n-1}}{a_2^2 a_3^2 \dots a_n^2} \right) - C_2 \frac{q_n}{a_{n+1}^2} \leq e^{-C_3 \tilde{t}^2 - C_4 \frac{q_n}{a_{n+1}^2}}, \quad (32)$$

where constants C_3 and C_4 are positive and do not depend on n .

Note that if $y = (y_1, y_2, \dots) \in Y$ then $\tilde{t} = a_2 \dots a_n \arg y_n$. Hence we can rewrite estimate (32) in the following way.

$$\hat{\mu}(y) \leq e^{-C_3 (a_2 \dots a_n \arg y_n)^2 - C_4 \frac{q_n}{a_{n+1}^2}}. \quad (33)$$

Fix n . The mapping $f : Y \rightarrow \mathbb{T}$ is defined by the formula $f(y) = \arg y_n$. It is easy to see that this mapping transform the Haar measure on Y into the Haar measure on \mathbb{T} . We use the

formula of the change of variable in the integral. Then

$$\int_{E_n} \hat{\mu}^k(y) dm_Y(y) \leq \int_{E_n} e^{-C_3 k(a_2 \dots a_n \arg y_n)^2 - C_4 k \frac{q_n}{a_{n+1}^2}} dm_Y(y) = \int_{-\pi/a_1 \dots a_n}^{\pi/a_1 \dots a_n} e^{-C_3 k(a_2 \dots a_n t)^2 - C_4 k \frac{q_n}{a_{n+1}^2}} dt. \quad (34)$$

Next, we change variable $s = a_2 \dots a_n t$. We have

$$\int_{-\pi/a_1 \dots a_n}^{\pi/a_1 \dots a_n} e^{-C_3 k(a_2 \dots a_n t)^2 - C_4 k \frac{q_n}{a_{n+1}^2}} dt = \frac{1}{a_2 \dots a_n} \int_{-\pi/a_1}^{\pi/a_1} e^{-C_3 k s^2 - C_4 k \frac{q_n}{a_{n+1}^2}} ds \leq C_5 \frac{1}{a_1 \dots a_n} \frac{e^{-C_4 k \frac{q_n}{a_{n+1}^2}}}{\sqrt{k}}, \quad (35)$$

where C_5 is a constant which does not depend on n and k .

Thus

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_{E_n} \hat{\mu}(y)^k dm_Y(y) \leq C_5 \sum_{n=1}^{\infty} \frac{1}{a_1 \dots a_n} \sum_{k=1}^{\infty} \frac{e^{-C_4 k \frac{q_n}{a_{n+1}^2}}}{\sqrt{k}}. \quad (36)$$

Since $\sum_{k=1}^{\infty} \frac{e^{-ak}}{\sqrt{k}} = O(\frac{1}{\sqrt{a}})$ (see e.g. [5]), we can continue inequality (36) in the following way

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \int_{E_n} \hat{\mu}(y)^k dm_Y(y) \leq C_6 \sum_{n=1}^{\infty} \frac{a_{n+1}}{a_1 \dots a_n \sqrt{q_n}}, \quad (37)$$

where C_6 is a constant which does not depend on n and k .

Arguing in the same way as at the end of proof of Theorem 1, we obtain that condition (26) is necessary for recurrence of a random walk defined by μ . ■

Remark 2. If we put in Theorem 2 all $a_j = p$ where p is prime, we obtain Theorem 4 of the article [5] for a random walk defined on the group H_p .

References

- [1] Darling, D. A., Erdos, P. On the recurrence of a certain chain, Proc. Amer. Math. Soc. 19 (1968), 336–338.
- [2] Dudley, R. M. Random walks on Abelian groups. Proc. Amer. Math. Soc. 13, (1962), 447–450.
- [3] G. M. Feldman, Functional Equations and Characterization Problems on Locally Compact Abelian Groups, EMS Tracts Math. 5, Eur. Math. Soc., Z"urich, 2008.
- [4] Flatto, L., Pitt, J. Recurrence criteria for random walks on countable Abelian groups. Illinois J. Math. 18, (1974), 1–19.
- [5] Freig, Nabil; Molchanov, S.A. On random walks on Abelian groups with infinite number of generators. (Russian) Vestn. Mosk. Univ., Ser. I 1978, No.5, 22-29 (1978).
- [6] E. Hewitt, and K.A. Ross, *Abstract Harmonic Analysis*, vol. 1, Springer-Verlag, Berlin, Gottingen, Heildelberg, 1963.

- [7] Kasimjanova, M.A. The recurrence of invariant Markov chains on a class of Abelian groups. (Russian) Vestn. Mosk. Univ., Ser. I 1981, No.3, 3-7 (1978).
- [8] Kesten, S., Spitzer, F. Random walk on countably infinite Abelian groups. Acta. Math. 114, (1965), 237–265.